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Real interpolation method on spaces of scalar integrable functions with respect to vector measures ☆

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ABSTRACT

For a given measurable space (Ω, Σ), and a vector measure $m : \Sigma \longrightarrow X$ with values in a Banach space X we consider the spaces of p-power integrable and weakly integrable, respectively, functions with respect to the measure m, $L^p(m)$ and $L^p_w(m)$, for $1 \leqslant p < \infty$. In this note we describe the real interpolated spaces that we obtain when the K-method is applied to any couple of these spaces.

1. Introduction

Let X be a Banach space and $m : \Sigma \longrightarrow X$ be a countably additive vector measure, where Σ is a σ-algebra of subsets of some nonempty set Ω. Associated with m are the Banach lattices $L^p(m)$ (and $L^p_w(m)$), with $1 \leqslant p < \infty$, of equivalence classes of functions $f : \Omega \longrightarrow \mathbb{R}$ (weakly) p-integrable with respect to m, equipped with the topology of convergence in p-mean. These spaces has been studied in [6]. Moreover, the two Calderón's complex interpolation spaces, $[X_0, X_1]_{[\theta]}$ and $[X_0, X_1]^{[\theta]}$, with $0 < \theta < 1$, of the complex Banach lattices couple (X_0, X_1), where X_0 and X_1 are the any of the spaces $L^p(m)$ or $L^p_w(m)$, with $1 \leqslant p < \infty$, were obtained in [7].

The purpose of this paper is to obtain the real interpolation spaces $(X_0, X_1)_{\theta,q}$, where $0 < \theta < 1 \leqslant q \leqslant \infty$, and X_0 and X_1 are, as above, $L^p(m)$ or $L^p_w(m)$, with $1 \leqslant p \leqslant \infty$. As it is well known, in the classical case, that is, when we are dealing with a positive finite measure instead of a vector measure, the Lorentz spaces appear in a natural way in this context. As far as we know, these spaces have not been considered or studied in the context of vector measures yet.

The results that we will obtain in this paper are quite different from those in the classical setting of a positive scalar measure. For a such measure μ it is well known that $L^p(\mu) = L^p_w(\mu)$, for all $1 \leqslant p \leqslant \infty$, and the classical interpolation result assures that $(L^1(\mu), L^\infty(\mu))_{\frac{1}{1-p},p} = L^p(\mu)$, with $1 < p < \infty$. For a vector measure m we shall see that $(L^1(m), L^\infty(m))_{\frac{1}{1-p},p} \subsetneq L^p(m) \subsetneq L^p_w(m)$, with $1 < p < \infty$. It seen to be that the intrinsic reason for these inequalities is that the inclusion $L^p(m) \subset L^p_w(m)$ can be strict and, what is more, these spaces can be non-reflexive spaces, even for $p > 1$, and on the other hand $L^1(m)$ or $L^1_w(m)$ can be reflexive.

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2. The spaces of p-integrable functions

In this section we present the basic definitions and results on integration with respect to vector measures that can be found in the recent monograph [11]. Let $m : \Sigma \longrightarrow X$ be a vector measure defined on a σ-algebra of subsets Σ of a nonempty set Ω; this will always mean that m is countably additive on Σ with values in a real Banach space X. We denote by X' its dual space, and by $X'' := (X')'$ its bidual. Also $B(X)$ denotes the unit ball of X. The semivariation of m is the set function $\|m\| : \Sigma \longrightarrow [0,\infty)$ defined by $\|m\|(A) := \sup\{|\langle m, x'\rangle|(A){:}\ x' \in B(X')\}$, $A \in \Sigma$, where $|\langle m, x'\rangle|$ is the total variation measure of the scalar measure $\langle m, x'\rangle$ given by $\langle m, x'\rangle(A) := \langle m(A), x'\rangle$, for all $A \in \Sigma$. Note that for a positive scalar (finite) measure m, the semivariation $\|m\|$ and the measure m coincide. Basic properties of the semivariation can be found in [5, Chapter IV, §10]. In particular, from [5, Lemma IV.10.5] we obtain the *continuity property* of the semivariation, namely: $\lim_{n\longrightarrow\infty} \|m\|(A_n) = 0$ for every sequence $(A_n)_n$ of measurable sets with $A_n \downarrow \varnothing$. A set $A \in \Sigma$ is called m-null if $\|m\|(A) = 0$. Two measurable real functions f and g defined on Ω are identified if they are equal m-a.e., that is, if $\{w \in \Omega{:}\ f(w) \neq g(w)\}$ is an m-null set. A measurable function $f : \Omega \longrightarrow \mathbb{R}$ is called *weakly integrable* (with respect to m) if $f \in L^1(|\langle m, x'\rangle|)$ for all $x' \in X'$. In this case (see [12, Corollary 3]) for each $A \in \Sigma$ there exists an element $\int_A f\,dm \in X''$ (called the *weak integral* of f over A) such that $\langle \int_A f\,dm, x'\rangle = \int_A f\,d\langle m, x'\rangle$ for all $x' \in X'$. The space $L^1_w(m)$ of all (equivalence classes of) weakly integrable functions becomes a Banach lattice when it is endowed with the natural order m-a.e., and the norm

$$\|f\|_1 := \sup\left\{ \int_\Omega |f|\,d\big|\langle m, x'\rangle\big|{:}\ x' \in B\big(X'\big) \right\}, \quad f \in L^1_w(m).$$

We say that a weakly integrable function f is *integrable* (with respect to m) if the vector $\int_A f\,dm \in X$ for all $A \in \Sigma$ (see [8,9] or [11]). The set $L^1(m)$ of all (equivalence classes of) integrable functions becomes an order continuous closed ideal of $L^1_w(m)$. In general, we have $L^1(m) \subsetneq L^1_w(m)$.

Now, if $1 < p < \infty$, we say that a measurable function $f : \Omega \longrightarrow \mathbb{R}$ *is weakly p-integrable* (with respect to m) if $|f|^p \in L^1_w(m)$, and *p-integrable* with respect to m if $|f|^p \in L^1(m)$. We denote by $L^p(m)$ the space of (equivalence classes of) p-integrable functions and by $L^p_w(m)$ the space of (equivalence classes of) weakly p-integrable functions. Obviously we have $L^p(m) \subseteq L^p_w(m)$. The natural norm for both spaces is given by

$$\|f\|_p := \sup\left\{ \left(\int_\Omega |f|^p\,d\big|\langle m, x'\rangle\big| \right)^{\frac{1}{p}}{:}\ x' \in B\big(X'\big) \right\}, \quad f \in L^p_w(m).$$

We know neither $L^p(m)$ nor $L^p_w(m)$ have to be reflexive spaces even if $p > 1$. See [6] for a detailed study of the relationship between the spaces $L^p(m)$ and $L^p_w(m)$. In particular, the inclusion $L^p_w(m) \subseteq L^1(m)$ holds for all $p > 1$. Moreover this embedding operator is weakly compact (see [6, Proposition 3.3]).

We also consider the space $L^\infty(m)$ of (equivalence classes of) essentially bounded functions (modulo m-a.e.) equipped with the supremum norm $\|\cdot\|_{L^\infty(m)}$. The inclusion $L^\infty(m) \subseteq L^1(m)$ holds and

$$\|f\|_{L^1(m)} \leqslant \|f\|_{L^\infty(m)} \|m\|(\Omega), \quad f \in L^\infty(m).$$

3. The distribution function and the decreasing rearrangement with respect to a vector measure

For a given measurable function $f : \Omega \longrightarrow \mathbb{R}$ we consider its *distribution function* $\|m\|_f : t \in [0,\infty) \longrightarrow \|m\|_f(t) \in [0,\infty)$, with respect to the vector measure m, defined by $\|m\|_f(t) := \|m\|(\{w \in \Omega{:}\ |f(w)| > t\})$, where $\|m\|$ is the semivariation of the measure m. This distribution function $\|m\|_f$ has similar properties that in the scalar case. In the next proposition we collect some of them that we will need in what follows. The proof is adapted from the scalar case that we can see in [3, Chapter 2, §1]. The furthermore part of its statement follows from the continuity property of the semivariation.

Proposition 1. *Suppose f, f_n $(n = 1, 2, \ldots)$ are measurable functions. The distribution function $\|m\|_f$ is bounded, decreasing, and right-continuous. Furthermore, if $|f_n| \uparrow |f|$ m-a.e., then $\|m\|_{f_n} \uparrow \|m\|_f$.*

For any scalar $s > 0$, note that $\inf\{t \geqslant 0{:}\ \|m\|_f(t) \leqslant s\} = \sup\{t \geqslant 0{:}\ \|m\|_f(t) > s\}$. Moreover $\sup\{t \geqslant 0{:}\ \|m\|_f(t) > s\} = \lambda(\{t \geqslant 0{:}\ \|m\|_f(t) > s\}) = \lambda_{\|m\|_f}(s)$, where $\lambda_{\|m\|_f}$ is the distribution function of $\|m\|_f$, with respect to the Lebesgue measure λ on the interval $[0,\infty)$.

Now consider the *decreasing rearrangement* of f (with respect to the measure m), defined for all $s > 0$, as the function

$$f_*(s) := \inf\big\{t \geqslant 0{:}\ \|m\|_f(t) \leqslant s\big\} = \lambda_{\|m\|_f}(s). \tag{1}$$

Thus we have a decreasing, and right-continuous function $f_* : s \in (0,\infty) \longrightarrow f_*(s) \in [0,\infty)$ such that $f_*(s) = 0$ for all $s \geqslant \|m\|(\Omega)$. In fact we may regard f_* as a function defined on the interval $(0, \|m\|(\Omega))$. Some properties linking the distribution function and the decreasing rearrangement are together in the next proposition.

Proposition 2. *Let f be a measurable function and $1 \leqslant p, q < \infty$. Then*:

1) $f_*(\|m\|_f(t)) \leqslant t$, *for all* $t \geqslant 0$.
2) $\|m\|_f(f_*(s)) \leqslant s$, *for all* $s > 0$.
3) $\lambda_{f_*} = \|m\|_f$, λ-*a.e., where* λ *is the Lebesgue measure.*
4) $\int_0^\infty t^{q-1}(\|m\|_f(t))^{\frac{q}{p}}\,dt = \frac{1}{p}\int_0^\infty (s^{\frac{1}{p}} f_*(s))^q \frac{ds}{s}$.
5) $\sup_{t>0} t^p \|m\|_f(t) = \sup_{s>0} s(f_*(s))^p$.

Proof. 1)–3) are straightforward.

4) This equality follows from [13, Theorem 8.7] by considering the generalized Young functions $\psi(t) = t^q$ and $\varphi(s) = s^{\frac{p}{q}}$. Finally a change of variable is needed.

5) It follows from 1) and 2) and the fact that both functions f_* and $\|m\|_f$ are decreasing. $\square$

Lemma 3. *Let f be a measurable function. Then*

$$\int_0^\infty \min\bigl\{\|m\|_f(s), t\bigr\}\,ds = \int_0^t f_*(s)\,ds, \quad t > 0. \tag{2}$$

Proof. It is not difficult to see that $\lambda_{\chi_{[0,t)}f_*} = t\chi_{[0,f_*(t))} + \lambda_{f_*\chi_{[f_*(t),\infty)}}$, for all $t > 0$. Now, taking into account [3, Proposition 2.1.8], we obtain

$$\begin{aligned}\int_0^t f_*(s)\,ds &= \int_0^\infty \chi_{[0,t)} f_*(s)\,ds = \int_0^\infty \lambda_{\chi_{[0,t)} f_*}(s)\,ds = \int_0^{f_*(t)} t\,ds + \int_{f_*(t)}^\infty \lambda_{f_*}(s)\,ds \\ &= \int_0^{f_*(t)} t\,ds + \int_{f_*(t)}^\infty \|m\|_f(s)\,ds = \int_0^\infty \min\bigl\{\|m\|_f(s), t\bigr\}\,ds. \qquad \square\end{aligned}$$

Remark 4. Note that the right-hand side in (2) can be infinite. In any case, since f_* is a decreasing and right-continuous function, the integral $\int_0^t f_*(s)\,ds$ is finite for all $t > 0$ if and only if there exists $t_0 > 0$ such that $\int_0^{t_0} f_*(s)\,ds < \infty$.

4. Lorentz spaces with respect to a vector measure

The interpolation space obtained by the K-method, applied to the spaces of p-power (weakly) integrable functions with respect to a vector measure $m : \Sigma \longrightarrow X$, will be formulated in terms of the family of the Lorentz spaces $L^{p,q}(\|m\|)$ with respect to the vector measure m which generalize the classical family of Lorentz spaces. Our first task therefore will be to define the Lorentz spaces and derive some of their elementary properties. For $1 \leqslant p, q \leqslant \infty$ the Lorentz space $L^{p,q}(\|m\|)$ with respect to the vector measure m consists of all measurable functions f for which the quantity

$$\|f\|_{L^{p,q}(\|m\|)} := \begin{cases} \left(\int_0^\infty (s^{\frac{1}{p}} f_*(s))^q \frac{ds}{s}\right)^{\frac{1}{q}} & (1 \leqslant q < \infty), \\ \sup_{s>0} s^{\frac{1}{p}} f_*(s) & (q = \infty) \end{cases}$$

is finite. The functional $f \longmapsto \|f\|_{L^{p,q}(\|m\|)}$ is not always a norm, even when $p, q \geqslant 1$. Triangle inequality fails because $\|m\|$ is not a measure (the semivariation is only subadditive). Nevertheless it is not difficult to prove (as in the scalar case), taking into account the inequality $(f+g)_*(s) \leqslant f_*(\frac{s}{2}) + g_*(\frac{s}{2})$, $s > 0$, that

$$\|f+g\|_{L^{p,q}(\|m\|)} \leqslant C(p,q)\bigl(\|f\|_{L^{p,q}(\|m\|)} + \|g\|_{L^{p,q}(\|m\|)}\bigr), \quad 1 \leqslant p, q \leqslant \infty,$$

where $C(p,q) = 2^{\frac{1}{p}}$ if $q = \infty$, and $C(p,q) = 2^{\frac{1}{p} - \frac{1}{q} + 1}$ if $1 \leqslant q < \infty$. And so $f \longmapsto \|f\|_{L^{p,q}(\|m\|)}$ is only a *quasi-norm*. However, as we shall see, with the help of real interpolation method, we will be able to prove that the spaces $L^{p,q}(\|m\|)$ are normable for $p > 1$ and $1 \leqslant q \leqslant \infty$.

The next result shows that, for any fixed p, the Lorentz spaces $L^{p,q}(\|m\|)$ increase as the secondary exponent q increases. However, inclusion relations among $L^{p,q}(\|m\|)$ spaces, with p varying, are like those for the Lebesgue spaces L^p of a positive finite measure, the secondary exponent q is not involved.

Proposition 5.

1) *Suppose $1 \leqslant p < \infty$ and $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$. Then $L^{p,q_1}(\|m\|) \subseteq L^{p,q_2}(\|m\|)$, and this inclusion is continuous.*
2) *Suppose $1 \leqslant p_1 < p_2 < \infty$ and $1 \leqslant q_1, q_2 \leqslant \infty$. Then $L^{p_2,q_2}(\|m\|) \subseteq L^{p_1,q_1}(\|m\|)$, and this inclusion is continuous.*

Proof. The proof is similar to the scalar case. □

In the special case in which $1 \leqslant p = q \leqslant \infty$, we denote the space $L^{p,p}(\|m\|)$ simply by $L^p(\|m\|)$. When the vector measure m is a finite positive scalar measure μ it is well known that $L^p(\|\mu\|)$ coincide isometrically with the Lebesgue space $L^p(\mu)$ for all $1 \leqslant p \leqslant \infty$. For a general vector measure m it is not difficult to see, as in the scalar case, that $L^\infty(\|m\|) = L^\infty(m)$, but, in the general case we don't have equality between $L^p(\|m\|)$ and $L^p(m)$. Let us consider first the next example.

Example 6. Define the vector measure m on the σ-algebra $\mathcal{P}(\mathbb{N})$ of all subsets of natural numbers by $m : A \in \mathcal{P}(\mathbb{N}) \longrightarrow m(A) := \sum_{n\in A} \frac{1}{n} e_n \in c_0$. Note that, for each $x' = (x'_n) \in \ell^1 = c'_0$, and each $A \subset \mathbb{N}$, the scalar measure (and its variation) associated to the functional x' are given by $\langle m, x'\rangle(A) = \sum_{n\in A} \frac{x'_n}{n}$ and $|\langle m, x'\rangle|(A) = \sum_{n\in A} \frac{|x'_n|}{n}$, respectively. On the other hand, as the measure m is positive, we have

$$\|m\|(A) = \big\|m(A)\big\|_{c_0} = \bigg\| \sum_{n\in A} \frac{1}{n} e_n \bigg\|_{c_0} = \frac{1}{\min(A)}, \quad A \in \mathcal{P}(\mathbb{N}). \tag{3}$$

Then, it is easy to verify that $L^1_w(m) = \{(f_n)_n\colon\ (n^{-1} f_n)_n \in \ell_\infty\}$ and $L^1(m) = \{(f_n)_n\colon\ (n^{-1} f_n)_n \in c_0\}$. On the other hand, from (3), we have

$$L^1\big(\|m\|\big) = \left\{ (f_n)_n\colon \int_0^\infty \frac{1}{\min\{n \in \mathbb{N}\colon\ |f_n| > t\}}\, dt < \infty \right\}.$$

It is clear that the function $f : \mathbb{N} \longrightarrow \mathbb{R}$, given by $(f_n)_n := (\frac{n+1}{\log(n+1)})_n$ is in $L^1(m)$, but let us verify that it does not belong to $L^1(\|m\|)$. For every $k = 1, 2, \ldots$, set $x_k := \frac{k+1}{\log(k+1)}$. Then, for every $N \geqslant 1$, we have

$$\begin{aligned} \int_0^\infty \frac{1}{\min\{n \in \mathbb{N}\colon\ |f_n| > t\}}\, dt &\geqslant \int_{x_1}^\infty \frac{1}{\min\{n \in \mathbb{N}\colon\ |f_n| > t\}}\, dt = \sum_{k=1}^\infty \int_{x_k}^{x_{k+1}} \frac{1}{k+2}\, dt \geqslant \sum_{k=1}^N \frac{x_{k+1} - x_k}{k+2} \\ &= -\frac{1}{\log 3} + \sum_{r=3}^{N+1} \frac{1}{r \log(r+1)} + \frac{N+3}{(N+2)\log(N+3)}, \end{aligned}$$

which clearly diverges as $N \longrightarrow \infty$, and therefore $f \notin L^1(\|m\|)$.

It is well known (see equalities (3.49) and (3.51) in [11]) that $L^p(m)$, with $1 \leqslant p < \infty$, is the $\frac{1}{p}$-power of the space $L^1(m)$. Similarly, the space $L^p(\|m\|)$, with $1 \leqslant p < \infty$, is the $\frac{1}{p}$-power of the space $L^1(\|m\|)$. Indeed, taking into account item 4) in Proposition 2, a measurable positive function f belongs to $L^p(\|m\|)$ if and only if the integral $\int_0^\infty t^{p-1} \|m\|_f(t)\, dt$ is finite. A simple change of variable shows that this happens if and only if the integral $\int_0^\infty \|m\|_{f^p}(t)\, dt$ is finite, which means that $f^p \in L^1(\|m\|)$. Then, according to [11, Lemma 2.20], $L^p(\|m\|) \subseteq L^p(m)$ if and only if $L^1(\|m\|) \subseteq L^1(m)$. Now we are going to prove this inclusion.

Proposition 7. *If $1 \leqslant p < \infty$, then*

$$L^{p,1}\big(\|m\|\big) \subseteq L^p\big(\|m\|\big) \subseteq L^p(m) \subseteq L^p_w(m) \subseteq L^{p,\infty}\big(\|m\|\big), \tag{4}$$

and these inclusions are continuous.

Proof. Note that, according to Proposition 5, we only have to prove the inclusions $L^p(\|m\|) \subseteq L^p(m)$ and $L^p_w(m) \subseteq L^{p,\infty}(\|m\|)$. Let us start with the first one. Taking into account [11, Lemma 2.20], it is enough to prove the inclusion $L^1(\|m\|) \subseteq L^1(m)$. Take a function $f \in L^1(\|m\|)$ and a functional x' in the unit ball of X'. Then, by applying [3, Proposition 2.1.8],

$$\int_\Omega |f|\, d\big|\langle m, x'\rangle\big| = \int_0^\infty \big|\langle m, x'\rangle\big|_f(t)\, dt \leqslant \int_0^\infty \|m\|_f(t)\, dt < \infty.$$

This means that $f \in L^1_w(m)$. To show that f is in $L^1(m)$ we have to prove that for every measurable subset A there exists an element $\int_A f\,dm \in X$ satisfying

$$\left\langle \int_A f\,dm, x' \right\rangle = \int_A f\,d\langle m, x'\rangle, \quad x' \in X'.$$

First we consider the case $A = \Omega$. Then, for each $n \in \mathbb{N}$ consider the measurable sets $A_n := \{w \in \Omega\colon |f(w)| \leqslant n\}$ and the bounded functions $f_n := f\chi_{A_n}$. Then $f_n \in L^1(m)$ for all $n \in \mathbb{N}$, and consequently $\int_\Omega f_n\,dm \in X$ for all $n \in \mathbb{N}$. We are going to see that $(\int_\Omega f_n\,dm)_n$ is a Cauchy sequence in X. In fact, using again [3, Proposition 2.1.8], we have for every $n \geqslant k$,

$$\begin{aligned}
\left\| \int_\Omega f_n\,dm - \int_\Omega f_k\,dm \right\|_X &= \left\| \int_\Omega (f_n - f_k)\,dm \right\|_X = \sup_{x' \in B(X')} \left| \left\langle \int_\Omega (f_n - f_k)\,dm, x' \right\rangle \right| \leqslant \sup_{x' \in B(X')} \int_\Omega |f_n - f_k|\,d\left|\langle m, x'\rangle\right| \\
&= \sup_{x' \in B(X')} \int_\Omega |f|\chi_{A_n \setminus A_k}\,d\left|\langle m, x'\rangle\right| = \sup_{x' \in B(X')} \int_0^\infty \left|\langle m, x'\rangle\right|_{f\chi_{A_n \setminus A_k}}(t)\,dt \\
&\leqslant \sup_{x' \in B(X')} \int_0^k \left|\langle m, x'\rangle\right|(A_n \setminus A_k)\,dt + \sup_{x' \in B(X')} \int_k^n \left|\langle m, x'\rangle\right|_f(t)\,dt \\
&\leqslant k\|m\|(A_n \setminus A_k) + \int_k^\infty \|m\|_f(t)\,dt \leqslant k\|m\|_f(k) + \int_k^\infty \|m\|_f(t)\,dt \longrightarrow 0,
\end{aligned}$$

as $k \longrightarrow \infty$. Note that the series $\sum_{k=1}^\infty \|m\|_f(k)$ is convergent because the function f is in $L^1(\|m\|)$, and so, the integral $\int_0^\infty \|m\|_f(t)\,dt$ is finite. Since $\|m\|_f$ is a decreasing function and the series is convergent, we have $k\|m\|_f(k) \longrightarrow 0$, as $k \longrightarrow \infty$. Therefore the sequence $(\int_\Omega f_n\,dm)_n$ converges in X and its limit $\int_\Omega f\,dm := \lim_n \int_\Omega f_n\,dm$ satisfies, for each $x' \in X'$ that

$$\left\langle \int_\Omega f\,dm, x' \right\rangle = \lim_n \left\langle \int_\Omega f_n\,dm, x' \right\rangle = \lim_n \int_\Omega f_n\,d\langle m, x'\rangle = \int_\Omega f\,d\langle m, x'\rangle.$$

Now, for a measurable set A and a function $f \in L^1(\|m\|)$, note that $f\chi_A \in L^1(\|m\|)$. To conclude apply just what we have proven to the function $f\chi_A$. Next we are going to prove the second inclusion, that is, $L^p_w(m) \subseteq L^{p,\infty}(\|m\|)$. Consider a function $f \in L^p_w(m)$. For each $t > 0$ we have $t\chi_{\{w \in \Omega\colon |f(w)| > t\}} \leqslant |f|$, and so $\|t\chi_{\{w \in \Omega\colon |f(w)| > t\}}\|_p \leqslant \|f\|_p$. But $\|t\chi_{\{w \in \Omega\colon |f(w)| > t\}}\|_p = t(\|m\|_f(t))^{\frac{1}{p}}$, and consequently $\sup_{t>0} t(\|m\|_f(t))^{\frac{1}{p}} \leqslant \|f\|_p$. According to 5) in Proposition 2, the function f is in $L^{p,\infty}(\|m\|)$. $\square$

Note that, for all $1 < p$ and $1 \leqslant q \leqslant \infty$, the inclusions $L^{p,q}(\|m\|) \subseteq L^1(\|m\|) \subseteq L^1(m) \subseteq L^1_w(m) \subseteq L^{1,\infty}(\|m\|)$ hold, and both $L^1(m)$ and $L^1_w(m)$ are Banach spaces.

5. Estimates for the *K*-functional

Let (X_0, X_1) be a couple of (quasi)Banach spaces with continuous inclusion $X_1 \subseteq X_0$. For each $t > 0$, the Peetre's K-functional is defined, for every $f \in X_0$, by

$$K(t, f) = K(t, f; X_0, X_1) := \inf\left\{\|f_0\|_{X_0} + t\|f_1\|_{X_1}\colon f_0 \in X_0,\ f_1 \in X_1,\ f = f_0 + f_1\right\}.$$

For $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$, the real interpolation space $(X_0, X_1)_{\theta,q}$ consists of all elements $f \in X_0$ having a finite (quasi)norm

$$\|f\|_{\theta,q} := \begin{cases} (\int_0^\infty (t^{-\theta}K(t,f))^q \frac{dt}{t})^{\frac{1}{q}} & (1 \leqslant q < \infty), \\ \sup_{t>0} t^{-\theta}K(t,f) & (q = \infty). \end{cases}$$

In that follows we consider a concrete case, namely: the spaces $X_0 = L^1(m)$ and $X_1 = L^\infty(m)$, where $m : \Sigma \longrightarrow X$ is a vector measure with values in the Banach space X. As usual, for two real numbers A and B, by $A \approx B$ we mean that $\frac{1}{c}A \leqslant B \leqslant cA$, and by $A \preccurlyeq B$ that $A \leqslant cB$, for some positive constant c independent of appropriate quantities. With this notation, for each function $f \in X_0$ and each $t > 0$, such that $K(t, f) < \infty$, there exists a measurable set $A := A(t, f) \in \Sigma$ such that

$$K(t, f) \approx \|f\chi_A\|_{X_0} + t\|f\chi_{\Omega \setminus A}\|_{X_1}. \tag{5}$$

To obtain this set $A(t,f)$, just take two functions $f_0 \in X_0, f_1 \in X_1$ for which $f = f_0 + f_1$ and $\|f_0\|_{X_0} + t\|f_1\|_{X_1} \leqslant 2K(t,f,X_0,X_1)$, and consider the set $A(t,f) := \{w \in \Omega\colon\ |f_0(w)| \geqslant |f_1(w)|\}$. In order to estimate the K-functional for a general function f it is convenient to recall that (in our cases) $K(t,f) \approx K(t,|f|)$ for every $t > 0$ and every function f. In that follows we suppose that $f \geqslant 0$ when we want to estimate the K-functional $K(t,f)$.

Proposition 8. *Let f be a function in $L^1(m)$. Then*

$$\sup_{s>0} s\min\{\|m\|_f(s), t\} \preccurlyeq K(t, f, L^1(m), L^\infty(m)), \quad t > 0.$$

In particular, $tf_*(t) \preccurlyeq K(t,f,L^1(m),L^\infty(m))$, for all $t > 0$.

Proof. Denote by $B(s) := \{w \in \Omega\colon\ f(w) > s\}$ for all $s > 0$, and observe that $\|m\|(B(s)) = \|m\|_f(s)$. Moreover $f \geqslant s\chi_{B(s)}$ for all $s > 0$, and so $K(t,f,L^1(m),L^\infty(m)) \succcurlyeq K(t, s\chi_{B(s)}, L^1(m), L^\infty(m)) = sK(t,\chi_{B(s)},L^1(m),L^\infty(m))$. From (5), there exists a measurable set $A(t,s) := A(t,\chi_{B(s)})$ such that

$$K(t, \chi_{B(s)}, L^1(m), L^\infty(m)) \succcurlyeq \|m\|(B(s) \cap A(t,s)) + t\|\chi_{B(s)\setminus A(t,s)}\|_{L^\infty(m)} \geqslant \min\{\|m\|(B(s)), t\}.$$

Now, multiply by s and take supremum to obtain

$$K(t, f, L^1(m), L^\infty(m)) \succcurlyeq \sup_{s>0} s\min\{\|m\|_f(s), t\} \tag{6}$$

which is the inequality of the statement. Finally, take the particular value $s := \frac{1}{2}f_*(t)$ in (6) we obtain $tf_*(t) \preccurlyeq K(t,f,L^1(m),L^\infty(m))$, for all $t > 0$, and this is what we want to prove. □

Remark 9. The same proof of Proposition 8 can be applied to a function f in $L^1_w(m)$ to obtain

$$\sup_{s>0} s\min\{\|m\|_f(s), t\} \preccurlyeq K(t, f, L^1_w(m), L^\infty(m)), \quad t > 0.$$

In particular, $tf_*(t) \preccurlyeq K(t,f,L^1_w(m),L^\infty(m))$, for all $t > 0$.

Proposition 10. *Let f be a function in $L^1(m)$. Then $K(t,f,L^1(m),L^\infty(m)) \preccurlyeq \int_0^\infty \min\{\|m\|_f(s),t\}\,ds$, $t > 0$.*

Proof. Given $t > 0$, suppose that $\int_0^\infty \min\{\|m\|_f(s),t\}\,ds < \infty$. In other case is nothing to prove. In that case note that the integral $\int_0^\infty \|m\|_f(s)\,ds$ is also finite, which is equivalent to say that $f \in L^1(\|m\|)$. Denote by $s_* := f_*(t)$, that is, $s_* = \inf\{s > 0\colon\ \|m\|_f(s) \leqslant t\}$, and observe that

$$\min\{\|m\|_f(s), t\} = t\chi_{[0,s_*)}(s) + \|m\|_f(s)\chi_{[s_*,\infty)}(s). \tag{7}$$

Now consider the nonnegative functions (recall that f is nonnegative)

$$f_0(w) := \begin{cases} 0, & f(w) \leqslant s_*, \\ f(w) - s_*, & f(w) > s_*, \end{cases} \quad \text{and} \quad f_1(w) := \begin{cases} f(w), & f(w) \leqslant s_*, \\ s_*, & f(w) > s_*. \end{cases}$$

Note that $f_1 \in L^\infty(m)$ and $\|f_1\|_{L^\infty(m)} \leqslant s_*$. On the other hand $f_0 \in L^1(m)$ because $0 \leqslant f_0 \leqslant f - s_*$ and $f \in L^1(m)$. Moreover, it is evident that $f = f_0 + f_1$. Finally, note also that $f_0(w) > s$ if and only if $f(w) > s + s_*$ and so, we obtain that $\|m\|_{f_0}(s) = \|m\|_f(s + s_*)$ for all $s > 0$, and consequently $\int_0^\infty \|m\|_{f_0}(s)\,ds < \infty$. In this way we deduce, taking into account (7) and the definition of the norm in $L^1(m)$, that

$$\begin{aligned} \|f_0\|_{L^1(m)} &= \sup\left\{\int_\Omega |f_0|\,d|\langle m,x'\rangle|\colon\ x' \in B(X')\right\} = \sup\left\{\int_0^\infty |\langle m,x'\rangle|_{f_0}(s)\,ds\colon\ x' \in B(X')\right\} \\ &\leqslant \int_0^\infty \|m\|_{f_0}(s)\,ds = \int_0^\infty \|m\|_f(s+s_*)\,ds = \int_0^{s_*} \|m\|_f(s+s_*)\,ds + \int_{s_*}^\infty \|m\|_f(s+s_*)\,ds \\ &\leqslant \int_0^{s_*} \|m\|_f(s_*)\,ds + \int_{s_*}^\infty \|m\|_f(s)\,ds \leqslant \int_0^{s_*} t\,ds + \int_{s_*}^\infty \|m\|_f(s)\,ds = \int_0^\infty \min\{\|m\|_f(s), t\}\,ds. \end{aligned}$$

On the other hand, using again (7), we have

$$t\|f_1\|_{L^\infty(m)} \leqslant ts_* = \int_0^{s_*} \min\bigl\{\|m\|_f(s), t\bigr\}\,ds \leqslant \int_0^{\infty} \min\bigl\{\|m\|_f(s), t\bigr\}\,ds.$$

Then

$$K\bigl(t, f, L^1(m), L^\infty(m)\bigr) \leqslant \|f_0\|_{L^1(m)} + t\|f_1\|_{L^\infty(m)} \leqslant 2\int_0^{\infty} \min\bigl\{\|m\|_f(s), t\bigr\}\,ds,$$

which is the inequality we want to obtain. □

Remark 11. The same proof of Proposition 10 can be applied to a function f in $L^1_w(m)$ to obtain $K(t, f, L^1_w(m), L^\infty(m)) \preccurlyeq \int_0^\infty \min\{\|m\|_f(s), t\}\,ds$, for all $t > 0$.

6. Some interpolation results

In this section we describe real interpolation results for the spaces of p-power (weakly) integrable functions with respect to a vector measure m. The starting point is the description of the interpolated space $(L^1(m), L^\infty(m))_{\theta,q}$, where $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$. The rest follows on the reiteration theorem together with a chain of inclusions about spaces of p-power (weakly) integrable functions that follows from Proposition 7. See (10) below in Remark 16.

Theorem 12. *Suppose* $0 < \theta < 1 \leqslant q \leqslant \infty$. *Then* $(L^1(m), L^\infty(m))_{\theta,q} = L^{\frac{1}{1-\theta},q}(\|m\|)$. *The equality also means equality as metric spaces, that is, the metrics are equivalent.*

Proof. Let f be a function in $L^1(m)$ and let $t > 0$. Suppose first that $q = \infty$. From Proposition 8 we obtain

$$\sup_{t>0} t^{1-\theta} f_*(t) = \sup_{t>0} t^{-\theta} t f_*(t) \preccurlyeq \sup_{t>0} t^{-\theta} K\bigl(t, f, L^1(m), L^\infty(m)\bigr). \tag{8}$$

Now, if $f \in (L^1(m), L^\infty(m))_{\theta,\infty}$, then the right-hand side in (8) is finite, and $f \in L^{\frac{1}{1-\theta},\infty}(\|m\|)$. Reciprocally, suppose that $f \in L^{\frac{1}{1-\theta},\infty}(\|m\|)$, in which case $\sup_{t>0} t^{1-\theta} f_*(t) < \infty$, and consequently $f_*(t) \preccurlyeq t^{\theta-1}$ for all $t > 0$. From Proposition 10 and Lemma 3 we know that

$$t^{-\theta} K\bigl(t, f, L^1(m), L^\infty(m)\bigr) \preccurlyeq t^{-\theta} \int_0^t f_*(s)\,ds \preccurlyeq t^{-\theta} \int_0^t s^{\theta-1}\,ds = \frac{1}{\theta}.$$

Then $\sup_{t>0} t^{-\theta} K(t, f, L^1(m), L^\infty(m))$ is finite and $f \in (L^1(m), L^\infty(m))_{\theta,\infty}$. In conclusion we have proved that $(L^1(m), L^\infty(m))_{\theta,\infty} = L^{\frac{1}{1-\theta},\infty}(\|m\|)$. Note that the inclusions between the spaces $(L^1(m), L^\infty(m))_{\theta,\infty}$ and $L^{\frac{1}{1-\theta},\infty}(\|m\|)$ are both continuous inclusions. Suppose now that $q < \infty$. In that case, from Proposition 10 and Lemma 3 we have $K(t, f, L^1(m), L^\infty(m)) \preccurlyeq \int_0^t f_*(s)\,ds$. Then, by applying Hardy's inequality [3, Lemma 3.3.9], with parameter $1 - \theta < 1$ and $1 \leqslant q < \infty$, to the nonnegative measurable function $f_*(s)$, we obtain

$$\|f\|^q_{\theta,q} = \int_0^\infty \bigl(t^{-\theta} K\bigl(t, f, L^1(m), L^\infty(m)\bigr)\bigr)^q \frac{dt}{t} \preccurlyeq \int_0^\infty \left(t^{-\theta} \int_0^t f_*(s)\,ds\right)^q \frac{dt}{t} \leqslant \frac{1}{\theta^q} \int_0^\infty \bigl(t^{1-\theta} f_*(s)\bigr)^q \frac{dt}{t}.$$

Now, from the definition of Lorentz spaces respect to a vector measure we deduce that $L^{\frac{1}{1-\theta},q}(\|m\|) \subseteq (L^1(m), L^\infty(m))_{\theta,q}$. Note that this inclusion is continuous. To obtain the opposite inclusion take into account the following inequality

$$t f_*(t) \preccurlyeq K\bigl(t, f, L^1(m), L^\infty(m)\bigr), \quad t > 0, \tag{9}$$

coming from Proposition 8. Then

$$\int_0^\infty \bigl(t^{1-\theta} f_*(t)\bigr)^q \frac{dt}{t} \preccurlyeq \int_0^\infty \bigl(t^{-\theta} K\bigl(t, f, L^1(m), L^\infty(m)\bigr)\bigr)^q \frac{dt}{t}.$$

From here it follows that $(L^1(m), L^\infty(m))_{\theta,q} \subseteq L^{\frac{1}{1-\theta},q}(\|m\|)$. Note also that the last inclusion is continuous. □

Corollary 13. *Suppose $0 < \theta < 1 \leqslant q \leqslant \infty$. Then $(L^1_w(m), L^\infty(m))_{\theta,q} = (L^1(m), L^\infty(m))_{\theta,q} = L^{\frac{1}{1-\theta},q}(\|m\|)$. The equalities are also true as metric spaces, that is, the metrics are equivalent.*

Proof. The equality for $q = \infty$ follows from Remarks 9 and 11 and the first part of the proof in the above theorem. The case $q < \infty$ follows from part (d) in [4, Theorem 3.4.2] because the closure of $L^\infty(m)$ in $L^1_w(m)$ is exactly the space $L^1(m)$. □

Corollary 14. *Suppose $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. Then $L^{p,q}(\|m\|)$ is a Banach lattice.*

Proof. From the corollary above we know that $L^{p,q}(\|m\|) = (L^1(m), L^\infty(m))_{1-\frac{1}{p},q}$, which is a Banach space. To obtain that the K-functional associated to the couple of Banach lattices $(L^1(m), L^\infty(m))$ is a lattice norm take into account the *decomposition property* of Banach lattices [1, Theorem 1.9]. □

Remark 15. Note that we cannot apply Corollary 13 to obtain the space $L^1(\|m\|)$ as an interpolated space. In fact, we don't know if $L^1(\|m\|)$ is normable.

Remark 16. Let $1 < p < \infty$ and take $\theta = 1 - \frac{1}{p}$. Putting $q = 1$ in the above Corollary 13 we obtain, in particular, $(L^1(m), L^\infty(m))_{\theta,1} = L^{p,1}(\|m\|)$. Similarly, we have $(L^1(m), L^\infty(m))_{\theta,\infty} = L^{p,\infty}(\|m\|)$ if we take $q = \infty$ in Corollary 13. Now from (4) in Proposition 7 we conclude that

$$\big(L^1(m), L^\infty(m)\big)_{\theta,1} \subseteq L^p(m) \subseteq L^p_w(m) \subseteq \big(L^1(m), L^\infty(m)\big)_{\theta,\infty}, \tag{10}$$

where $1 < p < \infty$ and $\theta = 1 - \frac{1}{p}$. In the terminology of [4, Theorem 3.5.2] the inclusions above say that the spaces $L^p(m)$ and $L^p_w(m)$ belong both to the both classes $\mathcal{C}_J(\theta, L^1(m), L^\infty(m))$ and $\mathcal{C}_K(\theta, L^1(m), L^\infty(m))$. For the cases $p = 1$ or $p = \infty$, that is, when we are considering the spaces $L^1(m)$ or $L^\infty(m)$, respectively, the above comments also apply because the space $L^1(m)$ is of the classes $\mathcal{C}_J(0, L^1(m), L^\infty(m))$ and $\mathcal{C}_K(0, L^1(m), L^\infty(m))$, and similarly $L^\infty(m)$ is of the classes $\mathcal{C}_J(1, L^1(m), L^\infty(m))$ and $\mathcal{C}_K(1, L^1(m), L^\infty(m))$. See [4, p. 49] just after Definition 3.5.1.

Corollary 17. *Suppose $0 < \eta < 1 \leqslant q \leqslant \infty$ and let $1 \leqslant p_0 < p_1 \leqslant \infty$. Then*

$$\big(L^{p_0}(m), L^{p_1}(m)\big)_{\eta,q} = \big(L^{p_0}_w(m), L^{p_1}(m)\big)_{\eta,q} = \big(L^{p_0}_w(m), L^{p_1}_w(m)\big)_{\eta,q} = L^{p,q}\big(\|m\|\big),$$

where $\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}$.

Proof. Having in mind (10) in the previous Remark, we can apply the reiteration theorem [4, Theorem 3.5.3] with parameters $\theta_0 = 1 - \frac{1}{p_0}$ and $\theta_1 = 1 - \frac{1}{p_1}$. Note that $\theta_0 = 0$ if $p_0 = 1$, and $\theta_1 = 1$ if $p_1 = \infty$. In any case, the reiteration theorem tells us that

$$\big(L^{p_0}(m), L^{p_1}(m)\big)_{\eta,q} = \big(L^{p_0}_w(m), L^{p_1}(m)\big)_{\eta,q} = \big(L^{p_0}_w(m), L^{p_1}_w(m)\big)_{\eta,q} = \big(L^1(m), L^\infty(m)\big)_{\theta,q},$$

where $\theta = (1-\eta)\theta_0 + \eta\theta_1$, in which case $1 - \theta = \frac{1}{p}$. Finally the above Corollary 13 gives $(L^1(m), L^\infty(m))_{\theta,q} = L^{\frac{1}{1-\theta},q}(\|m\|) = L^{p,q}(\|m\|)$, which is the last equality. □

Corollary 18. *The space $L^{p,q}(\|m\|)$ is reflexive for every $1 < p, q < \infty$.*

Proof. From Corollary 13 we know that $L^{p,q}(\|m\|) = (L^1(m), L^\infty(m))_{1-\frac{1}{p},q}$. By applying Maligranda and Quevedo result [10, Theorem 1] (see also Beauzamy's results in [2]) we obtain the reflexivity of the space $L^{p,q}(\|m\|)$ because the inclusion $L^\infty(m) \subseteq L^1(m)$ is weakly compact since $L^1(m)$ has order continuous norm. See also [6]. □

Remark 19. From the above result, it follows that $L^p(\|m\|)$ is a reflexive space for $1 < p < \infty$, but it is important to point out that the spaces $L^p(m)$ or $L^p_w(m)$ can be non-reflexive, even for $1 < p < \infty$. See [6]. In fact, if m is a vector measure such that $L^1(m) \neq L^1_w(m)$, then $L^p(m)$ and $L^p_w(m)$ are non-reflexive spaces for every $p > 1$.

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